Nahm's equations, configuration spaces and flag manifolds

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Abstract. We give a positive answer to the Berry-Robbins problem for any compact Lie group G, i.e. we show the existence of a smooth W-equivariant map from the space of regular triples in a Cartan subalgebra to the flag manifold G/T. This map is constructed via solutions to Nahm's equations and it is compatible with the $SO(3)$ action, where $SO(3)$ acts on G/T via a regular homomorphism from $SU(2)$ to G. We then generalize this picture to include an arbitrary homomorphism from $SU(2)$ to G. This leads to an interesting geometrical picture which appears to be related to the Springer representation of the Weyl group and the work of Kazhdan and Lusztig on representations of Hecke algebras.

Keywords: Berry-Robbins problem, Hecke algebras, Nahm's equations.

1 Introduction

This paper has an unusual origin, evolution and potential application. As explained in [2, 3] it arose from a problem posed by Berry and Robbins [6] in their investigation of the spin-statistics theorem. They asked the following simple question: is there, for each integer $n \geq 2$, a continuous map

$$f_n : C_n(\mathbb{R}^3) \to U(n)/T^n \tag{1.1}$$

compatible with the action of the symmetric group Σ_n? Here $C_n(\mathbb{R}^3)$ is the configuration space of n ordered distinct points of $\mathbb{R}^3$, and $U(n)/T^n$ is the well known flag manifold. The symmetric group acts freely on both spaces, by

permuting points in the first space and components of the flag in the second. For $n = 2$,

$$
\begin{aligned}
C_2(\mathbb{R}^3) &= \mathbb{R}^3 \times (\mathbb{R}^3 - 0) \\
U(2)/T^2 &= P_1(\mathbb{C}) = S^2
\end{aligned}
$$

and there is an obvious solution to (1.1). Note that this obvious solution is also compatible with the natural action of $SO(3)$ on both sides.

In [2] a positive answer was given to the Berry-Robbins question using an elementary construction of f_n. A more elegant construction was also proposed but this was dependent on the conjectured non-vanishing of a certain determinant. This question was pursued further in [3] and the conjecture has now been verified numerically for $n \leq 20$ [5].

The maps f_n of [2] are all compatible with the action of $SO(3)$, where we choose $SO(3)$ to act on $U(n)/T^n$ via the irreducible representation of $SU(2)$ on $\mathbb{C}^n$. This suggested a natural generalization of the Berry-Robbins question to other compact Lie groups G instead of $U(n)$. Let T be a maximal torus of G, then the Weyl group

$$
W = N(T)/T
$$

acts freely on the flag manifold G/T. Let $\mathfrak{h}$ be the Lie algebra of T, then W acts also on $\mathfrak{h}$ and on

$$
\mathfrak{h}^3 = \mathfrak{h} \otimes \mathbb{R}^3. \tag{1.2}
$$

The singular set Δ of this action on $\mathfrak{h}^3$ is the union of the codimension 3 subspaces which are the kernels of root homomorphisms

$$
\alpha \otimes 1 : \mathfrak{h}^3 \rightarrow \mathbb{R}^{3.}
$$

Then W acts freely on $\mathfrak{h}^3 - \Delta$ which is the space of **regular** triples in $\mathfrak{h}$ (i.e. with only $\mathfrak{h}$ as their common centralizer).

For $G = U(n)$ we recognise that G/T is the usual flag manifold and that

$$
\mathfrak{h}^3 - \Delta = C_n(\mathbb{R}^3).
$$

The obvious generalization of (1.1) is therefore to ask for a continuous map

$$
f_G : \mathfrak{h}^3 - \Delta \rightarrow G/T \tag{1.3}
$$

which is compatible with the action of the Weyl group. Again we can hope to find f_G which is also compatible with the action of $SO(3)$, where $SO(3)$ acts on

$h^3 - \Delta$, via the decomposition (1.2), and acts on G/T through some preferred homomorphism

$$\rho : SU(2) \to G. \tag{1.4}$$

There is a natural candidate for each compact Lie group G, generalizing the irreducible n-dimensional representation of $SU(2)$ for $U(n)$. This is given by the so-called **regular** (or **principal)** homomorphism ρ. This may be characterized by the fact that, after complexification, ρ takes the unipotent element of $SL(2, \mathbb{C})$ into a **regular** unipotent element of $G^{\mathbb{C}}(i.e.$ one which lies in a unique Borel subgroup). The regular homomorphism is unique up to conjugacy. Its action on G/T also factors through $SO(3)$.

It turns out that such a map f_G, with all the desired properties, can actually be extracted from previous work on **Nahm's equations** in [9] and this is Theorem B of section 3. The original purpose of this paper was to show how this comes about.

After the original solution of the Berry-Robbins problem in [2], various cohomological consequences were drawn in [4], and similar results were expected for other Lie groups. It was then suggested by Gus Lehrer that these ideas might be related to the Springer representation of the Weyl group and the extensive work done in this direction by Kazhdan and Lusztig (see for example [17] or [16]). This has led us to extend our investigations, using Nahm's equation, to include arbitrary homomorphisms ρ of $SU(2)$ into G. This leads to an interesting geometrical picture, generalizing the map (1.3), formulated as Theorem A in section 3. It is our hope that this will shed light on the work of Kazhdan and Lusztig and explain the geometry behind the Hecke algebras. It is perhaps worth pointing out that both Hecke algebras and Nahm's equations are related to physics, the former through the Jones polynomials of knots (and Witten's reinterpretation in terms of Chern-Simons theory) and the latter through the self-dual Yang-Mills equations and magnetic monopoles.

The paper is organised as follows. In §2 we review the key aspects of Nahm's equations and the various moduli spaces. Then in §3 we spell out the main construction which in particular gives the map (1.3) and we formulate Theorems A and B. In §4 we break the $SU(2)$-symmetry down to a circle subgroup and relate the geometry to that of the complex Lie group. In §5 we explain the relation of our construction to the Kazhdan-Lusztig work.

In order to keep the geometrical picture clear Sections 2-5 are presented in non-technical terms. The precise analytical details are then set out in section 6.

2   Nahm's equations and Lie groups

Since Nahm's equations will be our main technical tool it may be helpful to provide here a little background on how these equations first arose and what role in particular they play in Lie theory.

For any Lie algebra $\mathfrak{g}$ Nahm's equations are the system of 3 $\mathfrak{g}$-valued ordinary differential equations

$$\frac{dT_i}{dt} + [T_j, T_k] = 0 \qquad (2.1)$$

when (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and the T_i are functions of the real variable t.

While (2.1) makes sense for any Lie algebra these equations have a particularly simple interpretation when $\mathfrak{g}$ is the Lie algebra of a **compact** Lie group G (the case of interest to us). In this case we have a G-invariant metric $\langle, \rangle$ on $\mathfrak{g}$, enabling us to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. This leads to the well-known G-invariant skew 3-form ϕ on $\mathfrak{g}$ given by

$$\phi(T_1, T_2, T_3) = \langle T_1, [T_2, T_3] \rangle \qquad (2.2)$$

which also defines the bi-invariant (harmonic) exterior differential 3-form on G (unique up to a scalar for simple G). It is then easy to check that

**Nahm's equations are the gradient-flow
equations for ϕ as a function on $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$.** $\qquad (2.3)$

Regarding ϕ as a function on $\mathfrak{g} \otimes \mathbb{R}^3$ or on $\mathrm{Hom}(\mathbb{R}^3, \mathfrak{g})$ it is **invariant under the $SO(3)$-action on $\mathbb{R}^3$.**

To see this we observe that if

$$T : \mathbb{R}^3 \to \mathfrak{g}$$

is a linear map then

$$\Lambda^3 T : \Lambda^3 \mathbb{R}^3 \to \Lambda^3 \mathfrak{g}$$

sends the $SO(3)$-invariant oriented volume element of $\mathbb{R}^3$ to $T_1 \wedge T_2 \wedge T_3$.

In Lie theory it is standard to consider

$$\mathfrak{g} \oplus \mathfrak{g} \cong \mathfrak{g} \otimes \mathbb{R}^2 \cong \mathfrak{g}^{\mathbb{C}}$$

the complexified Lie algebra. What, one may ask, is the significance of replacing $\mathfrak{g} \otimes \mathbb{R}^2$ by $\mathfrak{g} \otimes \mathbb{R}^3$ as in Nahm's equations? The answer is that we should identify $\mathbb{R}^3$ here with the **imaginary quaternions**

$$\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H}).$$

To see why this is the case we should actually introduce a fourth Lie-algebra-valued function $T_0(t)$ and consider the expression

$$A = T_0 dt + T_1 dx_1 + T_2 dx_2 + T_3 dx_3 \qquad (2.4)$$

as defining a G-connection over

$$H = \mathbb{R}^4 = \mathbb{R} \oplus \mathrm{Im}\mathbb{H}.$$

Since the matrices in (2.4) depend only on t, and not on x, the natural gauge group to consider is simply the G-valued functions of t. Using the gauge freedom we can reduce T_0 to zero getting back to just 3 matrices T_1, T_2, T_3. More invariantly we should start with a G-bundle over $\mathbb{R}^4$ which has an action of the translations of $\mathbb{R}^3$. Then (2.4) describes a connection for this bundle which is $\mathbb{R}^3$-invariant and is written in an $\mathbb{R}^3$-invariant gauge. The matrices T_i then represent the difference between the Lie derivative and the covariant derivative in the ith direction, and are usually referred to as Higgs fields: they are infinitesimal automorphisms of the bundle.

Now in 4 dimensions we have the famous anti-self-duality (ASD) equations

$$*F = -F$$

where F is the curvature of a connection A. It was Donaldson [13] who first observed that, for the connection (2.4), and after gauging away T_0, **the** ASD **equations are identical with Nahm's equations.**

Now it is an important point that the ASD equations over $\mathbb{R}^4$ are, formally, the **hyperkähler moment map** for the action of the gauge group. This leads (formally) to a hyperkähler metric on moduli spaces of solutions. This observation is well-known to physicists as a consequence of super-symmetry and the concept of the hyperkähler quotient construction, developed in [15], was inspired by this.

Hyperkähler manifolds are Riemannian manifolds of dimension $4n$ with holonomy in $Sp(n)$, so that their tangent spaces are quaternionic. They have a 2-parameter family of complex structures (depending on an embedding $\mathbb{C} \to \mathbb{H}$, or on an imaginary quaternion I with $I^2 = -1$). They are the quaternionic counterparts of complex Kähler manifolds and they have twistor spaces in the sense of Roger Penrose.

All these general remarks apply, not only to the full four-dimensional ASD equations (where the matrices T_0, ..., T_4 in (2.4) depend on all 4 variables), but also to the (partially) translation invariant ones such as Nahm's equations (where the dependence is only on one variable). These moduli spaces of solutions to Nahm's equations should formally have hyperkähler metrics.

Of course appropriate boundary conditions need to be imposed and analytical details need to be checked. Originally Nahm introduced his equations in relation to non-abelian magnetic monopoles (which satisfy the Bogomolny equations, the $\mathbb{R}$-invariant version of the ASD equations) and the corresponding hyperkähler metrics were studied in detail in [1].

It was Kronheimer [20, 21] who first applied Nahm's equations to the study of Lie groups themselves, by altering the boundary conditions. For $SU(2)$-monopoles of charge n Nahm considered his equation for $G = U(n)$ on an

interval and took as boundary condition that the T_i had **simple regular poles** at each end. If the T_i have simple poles with residues σ_i then (2.1) shows that

$$\sigma_i = [\sigma_j, \sigma_k] \tag{2.5}$$

are the commutation relations (up to a factor 2) of the quaternions i, j, k

$$\sigma_1 = i/2, \ \sigma_2 = j/2, \ \sigma_3 = k/2$$

and thus are the standard generators of the Lie algebra of $SU(2)$. A pole is called regular if the n-dimensional representation of $\mathfrak{su}(2)$ given by the matrices (2.5) is irreducible.

In [21] Kronheimer considered poles of any type, characterized by an arbitrary homomorphism

$$\rho : \mathfrak{su}(2) \to \mathfrak{g}$$

given by the residues as in (2.5). By taking $\rho = 0$ at one end (so that the solution has no pole there), and a general ρ at the other, Kronheimer obtained as his moduli space a new hyperkähler manifold which (for almost all of its complex structures) could be identified with the nilpotent orbit in $\mathfrak{g}^{\mathbb{C}}$ corresponding to ρ (i.e. the one containing $\rho(x)$ where x is a nilpotent element of $\mathfrak{sl}(2, C)$ and ρ is understood as the complexification of (2.6)).

In [20] Kronheimer considered Nahm's equation on the half-line $t \geq 0$ and imposed finiteness at 0, and finite limits at ∞

$$T_i(t) \to T_i(\infty) = \tau_i,$$

where $T_1(\infty)$, $T_2(\infty)$, $T_3(\infty)$ are a regular (commuting) triple. For these boundary conditions (with the G-conjugacy class of the regular triple τ fixed) he found the moduli space to be a hyperkähler manifold which (for almost all of its complex structures) was a regular semi-simple orbit in $\mathfrak{g}^{\mathbb{C}}$.

These results of Kronheimer have since [11, 18] been extended to provide hyperkähler metrics for all complex co-adjoint orbits. This story is the quaternionic generalization of the complex Kähler metrics on co-adjoint orbits of G.

The moral of all this is the following. A compact (real) Lie group G has a complexification $G^{\mathbb{C}}$ with compact complex homogeneous spaces (e.g. $G^{\mathbb{C}}/B$) which have Kähler metrics. The Lie algebra $\mathfrak{g}$ has a (vector space) quaternionisation $\mathfrak{g} \otimes \mathbb{H}$, but there is no corresponding ''quaternionic group''. However the analogous ''homogeneous spaces'' do exist as hyperkähler manifolds. For many purposes $G^{\mathbb{C}}$ can be studied through, for example, the flag manifold $G^{\mathbb{C}}/B$, so we can view the hyperkähler structures on the complex co-adjoint orbits of $G^{\mathbb{C}}$ as substitutes for the non-existing quaternion group.

In this spirit the different homomorphisms $\rho : SU(2) \to G$ are the quaternionic analogous of 1-parameter subgroups $U(1) \to G$.

From this point of view Nahm's equation is the key to unlocking the "quaternionic nature of Lie groups". An area where this has proved its worth is in the clarification of the work of Brieskorn on Kleinian singularities (due to Kronheimer [19]) and its systematic extension to the Brieskorn-Grothendieck resolution of singularities of the nilpotent variety [24].

In [8, 9] other variants of the boundary conditions for Nahm's equations were studied. Here we shall be concerned with the equations on the half-line where, following Kronheimer, we take limiting regular triples at ∞ but as $t \to 0$ we impose a simple pole of type ρ. The case when ρ is the regular $SU(2)$ will give the construction of the map (1.3), while the other ρ will yield the more general picture to be discussed later.

3   The main construction

Let $\rho : su(2) \to \mathfrak{g}$ be a homomorphism, and consider solutions of Nahm's equations (2.1) on the half-line $0 < t < \infty$, with the boundary conditions:

(a)   there is a pole of type ρ as $t \to 0$

(b)   the T_i tend to a regular commuting triple in $\mathfrak{g}^3$ as $t \to \infty$.   (3.1)

We denote the space of such solutions by $N'(\rho)$. By taking the value at ∞ we get a map

$$N'(\rho) \to \mathfrak{g}^3. \tag{3.2}$$

Note that we have fixed a definite homomorphism ρ, but conjugate ρ give equivalent maps (3.2).

Now fix a maximal torus T of G and let $\mathfrak{h}$ be its Lie algebra. G acts on $\mathfrak{g}^3$ and on the regular commuting triples. Each orbit is of the form $G\tau$ where τ is a regular triple of $\mathfrak{h}$ and every orbit $G\tau$ meets $\mathfrak{h}^3$ in an orbit of the Weyl group W. We can therefore define a finite covering $N(\rho)$ of $N'(\rho)$ by the commutative diagram

$$\begin{array}{ccc} N(\rho) & \to & N'(\rho) \\ \downarrow & & \downarrow \\ \mathfrak{h}^3 - \Delta & \to & (\mathfrak{h}^3 - \Delta)/W \end{array} \tag{3.3}$$

where the vertical arrows assign to a solution of Nahm's equation its orbit type at ∞, arising from (3.2).

Fixing τ identifies $G\tau$ with G/T and hence, by taking the values at ∞, we get a natural map

$$\phi(\rho) : N(\rho) \to G/T. \tag{3.4}$$

$N(\rho)$ is a fibration over $\mathfrak{h}^3 - \Delta$ with fibre (at τ) $N(\rho, \tau)$ and the manifolds $N(\rho, \tau)$ are all hyperkähler. In fact if we denote by $M(\rho)$ the T-bundle over $N(\rho)$ induced by ϕ then $M(\rho)$ is a hyperkähler manifold and the map

$$\mu : M(\rho) \to \mathfrak{h}^3 - \Delta$$

is a hyperkähler moment map of the T-action. The manifolds $N(\rho, \tau)$ are just the **hyperkähler quotients.** $M(\rho)$ itself is also a suitable moduli space of solutions of Nahm's equations.

All these statements are best understood in terms of the gauged version of Nahm's equations involving the fourth matrix T_0. This, together with the more precise description of the analytical details will be explained in §6 .

The action of W on $N(\rho)$, implied by (3.3), is induced by an action of the normalizer $N(T)$ on $M(\rho)$. Moreover the group $SU(2)$ acts throughout, commuting with $N(T)$, and the map $\phi(\rho)$ of (3.4) is compatible with the $SU(2)$ action on G/T induced by ρ.

In fact all these constructions are compatible with yet another group. This is the group $Z(\rho)$, the centralizer of $\rho(SU(2))$ in G. Conjugation by an element of $Z(\rho)$ preserves the boundary conditions (3.1) and so induces an action on $N(\rho)$. The natural action of $Z(\rho)$ on G/T also commutes (by definition) with the action of $SU(2)$. Thus $Z(\rho)$ lifts also to an action on $M(\rho)$.

To sum up we can formulate our main result as

Theorem A.   *The asymptotic value of the solutions of Nahm's equations, with boundary conditions* (3.1)*, yield a diagram of maps*

$$
\begin{array}{ccc}
M(\rho) & \to & G \\
\downarrow & & \downarrow \\
N(\rho) & \xrightarrow{\phi} & G/T
\end{array}
\tag{3.5}
$$

and a compatible action of the group

$$N(T) \times SU(2) \times Z(\rho)$$

descending to an action of

$$W \times SU(2) \times Z(\rho)$$

for the bottom map ϕ.

For the fibre map

$$N(\rho) \to \mathfrak{h}^3 - \Delta \tag{3.6}$$

$W \times SU(2)$ acts naturally on the base, while $Z(\rho)$ acts trivially on the base but acts on the fibres $N(\rho, \tau)$. The torus T and the group $Z(\rho)$ both preserve the hyperkähler structure of $M(\rho)$, but $SU(2)$ rotates the complex structures.

There are three noteworthy special cases of ρ. These are

(a) $\rho = 0$. Then $Z(\rho) = G$ and, as will be discussed in the next section, $N(\rho, \tau)$ is the complexification $G^{\mathbb{C}}/T^{\mathbb{C}}$ of G/T. The map

$$\phi : G^{\mathbb{C}}/T^{\mathbb{C}} \to G/T$$

commutes with G. Observing that G/T sits inside $G^{\mathbb{C}}/T^{\mathbb{C}}$ with a contractible T-invariant slice it follows that ϕ must be a deformation retraction compatible with this G-action.

(b) ρ the regular $SU(2)$. Then $Z(\rho)$ is finite and, as will be shown in the next section, $N(\rho, \tau)$ is one point. Thus Theorem A, in this case, reduces to

Theorem B.   *The asymptotic value of the solutions of Nahm's equations, for conditions* (3.1) *with ρ regular, yields a map*

$$\phi : \mathfrak{h}^3 - \Delta \to G/T$$

compatible with $W \times SU(2)$.

This is the result, generalizing the case of $U(n)$, which arose from the Berry-Robbins paper and provided our original motivation.

(c) ρ the sub-regular[1] $SU(2)$. Then, as we shall see later, $N(\rho, \tau)$ is the 4-dimensional ALE space studied by Kronheimer [19]. In this case $Z(\rho)$ is finite for all simple G except $SU(n)$ when it is $U(1)$. This circular symmetry corresponds to the Gibbons-Hawking construction [14].

As mentioned in section 1, the essentials of the proof of Theorem A are already contained in [9], and the details are given in section 6.

4   The complex picture

Although not necessary for Theorem A this section explains the geometrical nature of the fibres $N(\rho, \tau)$ occurring in Theorem A. The complex story is also needed for the link with the work of Kazhdan and Lusztig outlined in section 5.

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[1]This means that the corresponding nilpotent orbit in $\mathfrak{g}^C$ is subregular (the unique codimension 2 orbit in the nilpotent variety).

In this section we shall break the symmetry of $\mathbb{R}^3$ by picking a preferred axis and consider the orthogonal projection

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2 \cong \mathbb{C},$$

identifying $\mathbb{R}^2$ with the complex plane. The symmetry group $SO(3)$ is then reduced to $SO(2) = U(1)$.

The preferred axis will pick out a distinguished complex symplectic structure on all the hyperkähler manifolds described in the preceding sections. We shall now analyse the complex manifolds that arise. In this we are essentially following Kronheimer [21] as extended by the second author [8, 7].

Let $\tau = (\tau_1, \tau_2, \tau_3)$ be a regular triple with projection $\pi(\tau) = \sigma = \tau_2 + i\tau_3$. We shall in the first instance assume that σ is a regular point of the complex Lie algebra $\mathfrak{h} \otimes \mathbb{C}$. Then the main result proved in [8] identifies the preferred complex symplectic structure of the manifold $N(\rho, \tau)$. To describe this we need to recall the slice $S(\rho)$ introduced by Slodowy [24]. First we extend ρ to a homomorphism of complex Lie algebras

$$\rho : \mathfrak{sl}(2, C) \to \mathfrak{g} \otimes C.$$

Let

$$h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis of $\mathfrak{sl}(2, C)$ and let H, X, Y be their images under ρ. We put

$$S(\rho) = Y + Z(X) \tag{4.1}$$

where $Z(X)$ is the centralizer of X in $\mathfrak{g} \otimes C$. Then $S(\rho)$ is a transverse slice to the orbit of Y. It is transverse to any adjoint orbit of $G^{\mathbb{C}}$ it meets. In particular it is transverse to the orbit $G^{\mathbb{C}}\sigma$ and so intersects this in a manifold. Then we have [8]

$$N(\rho, \tau) \cong G^{\mathbb{C}}\sigma \cap S(\rho) \tag{4.2}$$

where $N(\rho, \tau)$ is given its preferred complex structure. Varying τ_1, while keeping $\sigma = \tau_2 + i\tau_3$ fixed, gives different Kähler metrics to the complex manifold in (4.2).

If τ_1 is a regular point of $\mathfrak{h}$, then $(\tau_1, 0, 0)$ is a regular triple so that $N(\rho, \tau)$ is still a complex manifold for $\sigma = 0$, where the isomorphism (4.2) breaks down. To understand what happens here we have to explain the Brieskorn-Grothendieck theory of the simultaneous resolution.

Starting now with the complex Lie group $G^{\mathbb{C}}$ we let B be a Borel subgroup, $\mathfrak{b}$ its Lie algebra and $\mathfrak{h}^{\mathbb{C}}$ a Cartan subalgebra in $\mathfrak{b}$. The Grothendieck resolution is then given by the diagram

$$
\begin{array}{ccc}
G^{\mathbb{C}} \times_B \mathfrak{b} & \xrightarrow{\psi} & \mathfrak{g}^{\mathbb{C}} \\
\downarrow \theta & & \downarrow \chi \\
\mathfrak{h}^{\mathbb{C}} & \rightarrow & \mathfrak{h}^{\mathbb{C}}/W
\end{array}
\qquad (4.3)
$$

where B acts on $G^{\mathbb{C}}$ on the right and by the adjoint action on $\mathfrak{b}$. The vertical maps are given by taking the semi-simple parts (the ''eigenvalues''). The key property of this diagram is that the fibres of θ provide resolutions of the singularities of the fibres of χ and that θ is a smooth fibration (and topologically a product). Note in particular that $\theta^{-1}(0)$ is a resolution of the nilpotent variety $\mathfrak{N}$: it is isomorphic to the cotangent bundle $T^*(G^{\mathbb{C}}/B)$.

We can now restrict this diagram to the slice $S(\rho)$, giving the diagram

$$
\begin{array}{ccc}
\psi^{-1}(S(\rho)) & \rightarrow & S(\rho) \\
\downarrow \theta(\rho) & & \downarrow \chi(\rho) \\
\mathfrak{h}^{\mathbb{C}} & \rightarrow & \mathfrak{h}^{\mathbb{C}}/W
\end{array}
\qquad (4.4)
$$

Again the fibres of $\theta(\rho)$ resolve the singularities of the fibres of $\chi(\rho)$ and $\theta(\rho)$ is a smooth fibration. In particular the inverse image $\theta(\rho)^{-1}(0)$ resolves the singularities of $\mathfrak{N} \cap S(\rho)$.

The generic fibre of $\chi(\rho)$ is the manifold $G^{\mathbb{C}}(\sigma) \cap S(\rho)$ of (4.2). As shown in [9, 10] the manifold $\psi^{-1}S(\rho)$ of (4.4) can be naturally identified with the submanifold $N_{\tau_1}(\rho) \subset N(\rho)$ (with fixed regular τ_1). In other words the complex manifolds $N(\rho, \tau)$ are the fibres of $\theta(\rho)$ and in particular

$$
N(\rho; \tau_1, 0, 0) \qquad (4.5)
$$

is the Grothendieck resolution of $\mathfrak{N} \cap S(\rho)$.

Let us illustrate all this by examining the three special cases of ρ :

(a) $\rho = 0$, $S(\rho) = \mathfrak{g}^{\mathbb{C}}$, $N(0; \tau) = G^{\mathbb{C}}(\sigma)$ and $N(0; \tau_1, 0, 0)$ is the resolution of $\mathfrak{N}$ and diagram (4.4) is just (4.3).

(b) ρ the regular $\mathfrak{su}(2)$, $S(\rho)$ is a translate of $\mathfrak{h}^{\mathbb{C}}/W$, the manifold in (4.2) is just a point and $\theta(\rho)$ is an isomorphism.

(c) ρ the sub-regular $\mathfrak{su}(2)$, the manifolds in (4.2) have complex dimension 2 and the fibres of $\theta(\rho)$ are the ALE spaces as discussed by Kronheimer [19].

Considering again the general case, we have a fibration $N(\rho) \to h^3 - \Delta$ with hyperkähler manifolds $N(\rho, \tau)$ as fibres. The group $SU(2)$ acts on this fibration. The subgroup $U(1) = SO(2) \subset SO(3)$ fixing a direction of $\mathbb{R}^3$ has fixed points of the form $\tau = (\tau_1, 0, 0) \in h^3 - \Delta$ and so its double-cover $S \subset SU(2)$ acts on the fibre $N(\rho, \tau)$ over this point. As we have seen this fibre in its complex structure fixed by S is a complex manifold which can be identified with the resolution of $\mathfrak{N} \cap S(\rho)$. Thus $\mathfrak{N}(\rho, \tau)$ has a holomorphic action of $U(1)$, in addition to a commuting action of $Z(\rho)$. As this holomorphic action of $U(1)$ must leave Y fixed, it is the composition of the complex scalar action on $\mathfrak{g}^{\mathbb{C}}$ and of the adjoint action by $\rho(U(1))$. This will be explained more fully in section 6. The map

$$\mathfrak{N}(\rho, \tau) \to S(\rho)$$

defines a distinguished compact complex subspace which is the inverse image of the base point $Y \in S(\rho)$ (see (4.1)). From the Grothendieck resolution (4.4) we see that this is just **the fixed point set of the action of** $Ad(Y)$ on $G^{\mathbb{C}}/B = G/T$. Equivalently, viewing $G^{\mathbb{C}}/B$ as the space $\mathfrak{B}$ of all Borel subgroups, it is the set of all Borel subgroups whose Lie algebra contain Y. We shall denote it by $\mathfrak{B}_Y$. When ρ is the regular $SO(2)$, Y is regular and $\mathfrak{B}_Y$ is a point. When ρ is sub-regular, Y is sub-regular and $\mathfrak{B}_Y$ is 1-dimensional, consisting of rational curves intersecting as in the Dynkin diagram [24]. In general $\mathfrak{B}_Y \subset \mathfrak{N}(\rho, \tau)$ is the ''compact core'' of the open manifold, and carries all its topological information. More precisely, **the action of** S extends to an action of C^* **all of whose orbits have limits (as** $z \to \infty$) in $\mathfrak{B}_Y$. The observation essentially goes back to Slodowy [24] and will be recalled in detail in the next section.

5   Relation with Kazhdan-Lusztig

In a long series of papers (see [16, 17]) Kazhdan and Lusztig made an extensive study of representation of the Hecke algebras H associated to Weyl groups (both finite and affine). A comprehensive account of this theory is given in [12]. These algebras are defined over the finite Laurent series

$$A = \mathbb{C}[q, q^{-1}]$$

and reduce to the group algebras of the Weyl group when $q = 1$.

Kazhdan and Lusztig construct representations of H on the equivariant K - groups of certain subspaces of the flag manifold of the Lie group G. The purpose of this section is to show how all the ingredients in the Kazhdan-Lusztig construction arise naturally in our context. It is our hope that this will shed light on the geometric significance of the Hecke algebras. Essentially, by using the ''quaternionic'' aspect of Lie groups which we have been emphasizing we are able to move outside the purely complex theory of Lie groups where Kazhdan

and Lusztig work. Since they use the Grothendieck resolution (and ideas of Brieskorn and Slodowy) it is not surprising that the hyperkähler story described in previous sections should be relevant.

Given $\rho : SU(2) \rightarrow G$ we recall that we have the fibration (3.6)

$$N(\rho)$$
$$\downarrow$$
$$\mathfrak{h}^3 - \Delta$$

whose fibres $N(\rho, \tau)$ are hyperkähler manifolds, and that the group

$$W \times SU(2) \times Z(\rho)$$

acts on the fibration (where $Z(\rho)$ centralizes the image of ρ). We now fix a direction in $\mathbb{R}^3$ reducing the $SU(2)$ symmetry to a circle S. We identify the ring A with the character ring of S (over $\mathbb{C}$)

$$A = R(S) \otimes \mathbb{C} \qquad (5.1)$$

(since $S \subset SU(2)$ double-covers $SO(2) \subset SO(3)$, our q is the square-root of the one in [17].

This means that any space X on which S acts will have an equivariant K-group

$$K_S(X) \otimes \mathbb{C} \qquad (5.2)$$

which is an A-module. If X is not compact we shall use K-theory with **compact supports** in (5.2)

Consider now a fixed point τ for the action of S on $\mathfrak{h}^3 - \Delta$. If we choose our coordinates of $\mathbb{R}^3$ so that S defines rotation in the (x_2, x_3) plane then τ is fixed under S if it is of the form $(\tau_1, 0, 0)$. Note that the set of such points can be identified with the regular points of h and so the components are permuted by the Weyl group. A choice of component is essentially the same as a choice of Borel subgroup of $G^{\mathbb{C}}$ containing T, or equivalently a choice of complex structure on the flag manifold G/T.

The fibre $N(\rho, \tau)$ over τ has a complex structure (singled out by our choice of direction) and a holomorphic action of S. We can therefore consider the K-group (with compact support)

$$K_S(N(\rho, \tau)) \otimes C \qquad (5.3)$$

Inside $N(\rho, \tau)$ we have its ''compact core'', namely the fixed-point set $\mathfrak{B}_Y$ of the nilpotent element $Y \in \mathfrak{sl}(2, \mathbb{C})$, as explained in §4, and the action of $\mathbb{C}^*$ (complexification of S) has all limits $z \rightarrow \infty$ in $\mathfrak{B}_Y$.

Now Kazhdan-Lusztig work with the ''homology'' version K_0^S of K_S^0 and observe that, in the situation just described, we have a natural isomorphism

$$K_0^S(\mathfrak{B}_Y) \otimes C \cong K_S^0(N(\rho, \tau)) \otimes \mathbb{C}. \qquad (5.4)$$

It is modules such as these in (5.4) (and various refinements) that are the A-modules studied by Kazhdan and Lusztig. One obvious refinement is to enhance the symmetry from S to $S \times Z(\rho)$, or to a subgroup of this.

The Weyl group W does not act on the spaces in (5.4), it permutes them. However we also have the map

$$\phi : N(\rho) \to G/T$$

defined by (3.4) and this is compatible with the action of $W \times Z(\rho)$. This makes the groups in (5.4) into modules over

$$K_S(G/T) \otimes \mathbb{C}$$

(where S acts on G/T via ρ) and more generally we can replace S by $S \times Z(\rho)$.

Let us now describe, somewhat speculatively, why this picture might help to explain the geometric significance of the Hecke algebra and its modules. As we have seen the K -groups in question, disregarding for the moment the S-equivariance, are K-groups of fibres over $\mathfrak{h}^3 - \Delta$ with an action of W on the fibration. Alternatively they are K-groups of fibres over $(\mathfrak{h}^3 - \Delta)/W$. In a non-equivariant situation this gives rise to the monodromy action of W. The action of W on the homology of the fibres essentially gives the Springer representations. In an equivariant situation (e.g. with an S -action) it is not clear what replaces monodromy, since S only acts on fibres over its fixed points. This suggests that $(\mathfrak{h}^3 - \Delta)/W$, together with its S-action, somehow produces the Hecke algebra (instead of the fundamental group) and that bundles over this space (together with compatible S-action) yield H-modules. One small piece of evidence in favour of this idea is to note that the S-equivariant analogue of a path from a point τ (fixed by S) to its transform $\omega(\tau)$, $\omega \in W$, is a 2-sphere acted on by S. The equivariant K -theory of such a 2-sphere is an A-module with one generator, satisfying a quadratic equation which is essentially the defining equation for generators of H.

Unfortunately, although this is an appealing idea, we have not yet seen how to carry it out. What we have done however is to put the general Kazhdan-Lusztig construction into a more natural form.

In particular the circle symmetry is enlarged to a full $SU(2)$-action. We hope the pay-off will emerge later.

6   Analytic details

In this section we shall explain the analytic details behind the main construction leading to the proof of Theorem A and in particular show how to define the spaces $N(\rho)$ and $M(\rho)$ of section 3 as moduli spaces of solutions to Nahm's equations. The Nahm equations will be the full translation-invariant anti-self-duality equations on $\mathbb{R}^4$:

$$\dot{T}_i + [T_0, T_i] + [T_k, T_j] = 0 , \qquad (6.1)$$

where (i, j, k) run over cyclic permuations of $(1, 2, 3)$. This form of Nahm's equations admits an action by the gauge group of G-valued functions $g(t)$:

$$
\begin{aligned}
T_0 &\mapsto \mathrm{Ad}(g)T_0 - \dot{g}g^{-1} \\
T_i &\mapsto \mathrm{Ad}(g)T_i , \qquad i = 1, 2, 3.
\end{aligned}
\qquad (6.2)
$$

The component T_0 can be gauged away if we allow arbitrary gauge transformations. We recall that the space $N'(\rho)$ was defined as the space of solutions to Nahm's equations on the half-line with poles of type ρ at $t = 0$ and approaching a regular commuting triple as $t \to +\infty$. As Kronheimer [20] observes such a solution must approach its limit exponentially fast.

Let Ω be the space of exponentially fast decaying functions in $C^1[0, +\infty]$, i.e.:

$$\Omega = \left\{ f : (0, \infty] \to \mathfrak{g}; \exists_{\eta>0} \sup_{t \geq 0} \left(e^{\eta t} \|f(t)\| + e^{\eta t} \|df/dt\| \right) < +\infty \right\}. \quad (6.3)$$

To define $N(\rho)$ let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and consider solutions to (6.1) on the half-line satisfying the following boundary conditions at infinity:

(i) $T_0(+\infty) = 0$;

(ii) $T_i(+\infty) \in \mathfrak{h}$ for $i = 0, \ldots, 3$;

(iii) $(T_1(+\infty), T_2(+\infty), T_3(+\infty))$ is a regular triple, i.e. its centralizer is $\mathfrak{h}$;

(iv) $(T_i(t) - T_i(+\infty)) \in \Omega$ for $i = 0, 1, 2, 3$.

In addition, the boundary conditions at $t = 0$ are the same as for $N'(\rho)$. This space is acted upon by the gauge group $\mathcal{G}$ whose Lie algebra consists of *bounded* C^2-paths $\rho : [0, +\infty) \to \mathfrak{g}$ with $\rho(0) = 0$ and $\dot{\rho}, [\tau, \rho]$ both belonging to Ω for any regular element τ of $\mathfrak{h}$. This means that any element of $\mathcal{G}$ is asymptotic to an element of $T = \exp \mathfrak{h}$. Observe that we have a free action of $W = N(T)/T$ on $N(\rho)$ given by gauge transformations asymptotic to elements of $N(T)$.

We claim that the moduli space we obtain is the space $N(\rho)$ defined by the diagram (3.3). Indeed, we see that we can always make T_0 identically zero

via a gauge transformation $g(t)$ with $g(0) = 1$. This gives us a projection $N(\rho) \to N'(\rho)$. Now suppose we have two solutions (T_i) and (T_i') in $N(\rho)$ which map to the same element of $N'(\rho)$. This means that (T_i) and (T_i') are gauge equivalent via a gauge transformation $g(t)$ with $g(0) = 1$. Moreover, as the limit of both (T_i) and (T_i') is a regular triple in the same Cartan subalgebra, $g(t)$ is asymptotic to an element of $N(T)$ and so (T_i) and (T_i') are in the same W-orbit.

The manifold $N(\rho)$ is not a hyperkähler. Nevertheless it is fibred by the hyperkähler manifolds $N(\rho, \tau)$ defined by fixing the limit $\tau = (\tau_1, \tau_2, \tau_3)$ of T_1, T_2, T_3 [8] (this is the fibration defined in (3.4)). As pointed out in section 2, a moduli space of solutions to Nahm's equations is expected to carry a hyperkähler structure if it can be (formally) realised as a an infinite-dimensional hyperkähler quotient. The spaces $N(\rho, \tau)$ are such quotients of the flat affine manifold consisting of all functions (T_0, T_1, T_2, T_3) with prescribed boundary conditions.

When $\sigma = \tau_2 + i\tau_3$ is a regular element of $\mathfrak{g}^{\mathbb{C}}$, $N(\rho, \tau)$ has the complex structure (corresponding to choosing the x_1-axis in $\mathbb{R}^3$) described in (4.2). In general, a complex structure of a hyperkähler moduli space of solutions to Nahm's equations can be identified by writing Nahm's equations as equations for $\mathfrak{g}^{\mathbb{C}}$-valued functions. If we choose an isomorphism (compatible with the usual metrics) $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$, i.e. we choose complex coordinates, say $(t + ix_1, x_2 + ix_3)$, on $\mathbb{R}^4$, we can put

$$\alpha := T_0 + iT_1 , \ \beta := T_2 + iT_3$$

The Nahm equations can then be written as:

$$\frac{d}{dt}(\alpha + \alpha^*) + [\alpha, \alpha^*] + [\beta, \beta^*] = 0 \tag{6.4}$$

and

$$\frac{d}{dt}\beta = [\beta, \alpha] \tag{6.5}$$

The second equation is preserved by the complex gauge transformations and our moduli space as a complex (in fact complex-symplectic) manifold is just

$$\big(\text{solutions to (6.5)}\big) / \big(\text{complex gauge transformations}\big).$$

This is an example of identifying hyperkähler and complex symplectic quotients [15].

Returning to $N(\rho, \tau)$, we first observe, after Kronheimer [20], that when $\rho = 0$, $N(\rho, \tau)$ is the complex adjoint orbit of σ with the holomorphic identification given by

$$(\alpha(t), \beta(t)) \mapsto \beta(0). \tag{6.6}$$

For a general ρ, $N(\rho, \tau)$ can be defined as the hyperkähler quotient of the product manifold $N(0, \tau) \times N_\rho$, where N_ρ is the moduli space of solutions to Nahm's equations on the interval $(0, 1]$ with poles of type ρ at $t = 0$ and regular at $t = 1$ (mod gauge transformations which are 1 at both endpoints). This hyperkähler manifold has been studied in detail in [8] where it was shown that with respect to any complex structure it is $S(\rho) \times G^{\mathbb{C}}$ ($S(\rho)$ is the transversal slice defined in (4.1)). In particular, when $\rho = 0$, N_ρ is isomorphic to $T^* G^{\mathbb{C}}$ as a complex-symplectic manifold [22, 8].

Both $N(0, \tau)$ and N_ρ admit a hyperkähler G-action given by gauge transformations with arbitrary values at $t = 0$ and $t = 1$, respectively. Taking the hyperkähler quotient of $N(0, \tau) \times N_\rho$ by the diagonal G is equivalent to gluing the solutions in N_ρ at $t = 1$ to those in $N(0, \tau)$ at $t = 0$, and so it results in the manifold $N(\rho, \tau)$. On the other hand, the complex symplectic quotient of $\big(S(\rho) \times G^{\mathbb{C}}\big) \times \mathcal{O}(\sigma)$ by $G^{\mathbb{C}}$ is easily seen to be $S(\rho) \cap \mathcal{O}(\sigma)$ (the complex moment map on $S(\rho) \times G^{\mathbb{C}}$ is $\mu(\beta, g) = \mathrm{Ad}(g)\beta$ and on $\mathcal{O}(\sigma)$ it is the identity). The general mantra of identifying hyperkähler and complex-symplectic quotients gives[2] us the complex structure of $N(\rho, \tau)$.

$N(\rho)$ admits an action of $SU(2)$ defined as follows. Let A be an element of $SU(2)$. Then A acts on $N(\rho)$ by rotating the ''vector'' $(T_1(t), T_2(t), T_3(t))$ and then acting on the resulting solution to Nahm's equations with a gauge transformation equal to $\rho(A)^{-1}$ at $t = 0$. This action leaves invariant the residues of (T_0, T_1, T_2, T_3) at $t = 0$.

We shall now explain the diagram (3.5) of Theorem A in terms of the solutions to Nahm's equations. We shall define a torus bundle $M(\rho)$ over $N(\rho)$ which will be a hyperkähler manifold (more exactly: a hypercomplex manifold with a compatible symmetric form which is generically non-degenerate). To define this torus bundle we first observe that $N(\rho)$ can be also defined as $\mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is defined by omitting the condition (i) on the solutions to Nahm's equations in the definition of $N(\rho)$ and the gauge group is enlarged to $\mathcal{G}$ consisting of paths $g(t)$ asymptotic to $\exp(ht + \lambda h)$ for some $h \in \mathfrak{h}$ and $\lambda \in \mathbb{R}$. In other words the Lie algebra of $\mathcal{G}$ consists of C^2-paths $\rho : [0, +\infty) \to \mathfrak{g}$ such that

  (i) $\rho(0) = 0$ and $\dot{\rho}$ has a limit in $\mathfrak{h}$ at $+\infty$;

  (ii) $(\dot{\rho} - \dot{\rho}(+\infty)) \in \Omega$, and $[\tau, \rho] \in \Omega$ for any regular element $\tau \in \mathfrak{h}$;

The torus bundle $M(\rho)$ over $N(\rho)$ is defined as the quotient $\mathcal{A}/\mathcal{G}_0$, where $\mathcal{G}_0$ is defined as $\mathcal{G}$ with the added condition:

  (iii) $\lim_{t \to +\infty}(\rho(t) - t\dot{\rho}(+\infty)) = 0$.

---

[2]Strictly speaking a hyperkähler quotient can in general only be identified with an open subset of a complex-symplectic quotient (of semi-stable points). The analytic argument that in our case the two coincide is given in [8].

In other words, elements $g(t)$ of $\mathcal{G}_0$ are asymptotic to $\exp(ht)$ for some $h \in \mathfrak{h}$. It is clear that $\mathcal{G}/\mathcal{G}_0 = \exp(\mathfrak{h})$ and therefore $M(\rho)$ is a torus bundle over $N(\rho)$.

We observe that this $M(\rho)$ is the one defined by the diagram (3.5). Indeed, we can make T_0 identically zero by a gauge transformation asymptotic to $g \exp(ht)$ where $g \in G$ and $h \in \mathfrak{h}$. Since we quotient by $\mathcal{G}_0$ we obtain a well defined element of G fitting into the diagram (3.5) (observe that in the above description of $N(\rho)$, we obtain a gauge transformation asymptotic to $g \exp(ht)$ but defined only up to the action of T).

The hyperkähler structure of $M(\rho)$ is part of the general story discussed in section 2 and its existence is proved in detail in [9]. In particular the hyperkähler moment map for the action of T on $M(\rho)$ is given by $(T_1(+\infty), T_2(+\infty), T_3(+\infty))$, and so the hyperkähler quotients are the fibers $N(\rho, \tau)$ of the map (3.6). There is an action of $SU(2)$ defined on $M(\rho)$ in exactly the same way as for $N(\rho)$. This action rotates the complex structures of $M(\rho)$ which are therefore all equivalent. $M(\rho)$ as a complex manifold is discussed at length in [9]. In the complex picture only the action of the $U(1) \subset SU(2)$ preserving the chosen complex structure is visible. This is the action described towards the end of §4.

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